Credit Ratings
The Debt Leverage Ratio

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June, 2023

It should be no surprise that with higher debt leverage ratios come lower credit ratings. For example, Intel with a debt leverage ratio of 5% should have a credit rating that is much higher then Intel with a debt leverage ratio of 80%. Given that a company's financial structure is fixed, credit ratings are a function of debt leverage.

In this white paper we will build a model to estimate the debt leverage ratio that is consistent with a given credit rating. To that end will will work throught the following hypothetical problem...

Our Hypothetical Problem

We are given the following model parameters for ABC Company...

Table 1: Model Parameters

Symbol	Description Value	
A_0	Asset value at time zero	10,000,000
κ	Asset return - mean	0.0953
σ	Asset return - volatility	0.3500
ϕ	Annual dividend yield	0.0513
β	Default point factor	0.9000
T	Debt term in years	5.0000

We are tasked with filling out the following table (default point and debt leverage ratio)...

Table 2: Credit Ratings [1]

Credit Rating	Default Rate	Default Point	Leverage Ratio
AAA	0.04%	?	?
AA	0.11%	?	?
A	0.28%	?	?
BBB	0.51%	?	?
BB	1.69%	?	?
B	3.34%	?	?

We will define the variable S_t to be the survival rate (probability that debt does not default) over the time interval $[0, t]$, the variable λ to be the hazard rate (annual default rate from the table above), and the variable t to be time in years. The equation for the cumulative default rate at time t is... [1]

$$\text{if... } S_t = \text{Exp}\left\{ -\lambda\, t \right\} \text{ ...then... Cumulative default rate} = 1 - S_t = 1 - \text{Exp}\left\{ -\lambda\, t \right\} \tag{1}$$

Building Our Model

We will define the variable A_t to be asset value (i.e. enterprise value) at time t, the variable κ to be the weighted-average cost of capital, the variable ϕ to be the dividend yield, the variable σ_A to be asset return volatility, and the variable z to be a normally-distributed random variable with mean zero and variance one. The equation for random asset value at time t is...

$$A_t = A_0 \,\mathrm{Exp}\left\{ \left(\kappa - \phi - \frac{1}{2}\,\sigma_A^2 \right) t + \sigma_A \sqrt{t}\, z \right\} \text{ ...where... } z \sim N\left[0,1\right] \tag{2}$$

We will define the variable m to be asset return mean (capital gains) under the actual probability Measure P and the variable v to be asset return variance. The equations for return mean and variance over the time interval $[0,t]$ are...

$$m = \left(\kappa - \phi - \frac{1}{2}\,\sigma_A^2 \right) t \text{ ...and... } v = \sigma_A^2 t \tag{3}$$

Using the definitions in Equation (3) above, we can rewrite random asset value Equation (2) above as...

$$A_t = A_0 \,\mathrm{Exp}\left\{ \theta \right\} \text{ ...where... } \theta \sim N\left[m,v\right] \tag{4}$$

We will define the variable D_t to be debt balance at time t (i.e. debt principal plus accrued interest). If we assume that the dividend yield in Equation (2) above is sufficient to pay interest on the debt over the time interval $[0,t]$ then we can make the following statement...

$$D_0 = D_t \tag{5}$$

We will define the variable Γ to be the ratio of debt to assets (i.e. leverage ratio). Using Equation (5) above, the equation for the leverage ratio is...

$$\Gamma = \frac{D_0}{A_0} \text{ ...such that... } D_0 = \Gamma A_0 \tag{6}$$

We will define the variable a to be the default point, which is defined as the random return over the time interval $[0,t]$ that results in asset value at time t equal to debt value at time t, and the variable β to be the default point factor. Using Equations (4), (5) abd (6) above, the equation for the default point is...

$$\text{if... } A_0 \,\mathrm{Exp}\left\{ a \right\} = \beta\, D_t = \beta\, \Gamma A_0 \text{ ...then... } a = \ln(\beta\, \Gamma) \tag{7}$$

We will define the function $f(a)$ to be the probability that the borrower will default on the debt at time t. Using the probability density function of the normal distribution and the equations above, the equation for the cumulative probability of default over the time interval $[0,t]$ is... [3]

$$f(a) = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi\,v}}\, \mathrm{Exp}\left\{ -\frac{1}{2\,v}\,(\theta - m)^2 \right\} \delta\theta \tag{8}$$

The equation for the derivative of Equation (8) above with respect to the default point a is...

$$f'(a) = \frac{\delta f(a)}{\delta a} = \sqrt{\frac{1}{2\,\pi\,v}}\, \mathrm{Exp}\left\{ -\frac{1}{2\,v}\,(a - m)^2 \right\} \tag{9}$$

We will define the variable a to be the actual default point and the variable $\hat{a}$ to be the estimated default point. To solve for the default point a, we will use the Newton-Raphson method for solving non-linear equations. Our equation to iterate is... [2]

$$\text{Iterate... } a + \hat{\epsilon} = \hat{a} + \frac{f(a) - f(\hat{a})}{f'(\hat{a})} \text{ ...until... } \hat{\epsilon} \approx 0 \text{ ...and... } \hat{a} \approx a \tag{10}$$

Once we have derived the default point a via Equation (10) above, using Equation (7) above the equation for debt leverage that is consistent with a given credit rating is...

$$\text{if... } a = \ln(\beta\, \Gamma) \text{ ...then... } \Gamma = \frac{1}{\beta}\, \mathrm{Exp}\left\{ a \right\} \tag{11}$$

The Answer To Our Hypothetical Problem

Using Equation (3) above and the data in Table 1 above, the return mean and variance over the time interval $[0, T]$ are...

$$m = \left(0.0953 - 0.0513 - \frac{1}{2} \times 0.35^2\right) \times 5.00 = -0.0862 \text{ ...and... } v = 0.35^2 \times 5.00 = 0.6125 \tag{12}$$

The answer to our hypothetical problem is the table below...

Table 3: Leverage Ratios

Credit Rating	Default Rate	Default Point	Leverage Ratio
AAA	0.04%	-2.34	10.73%
AA	0.11%	-2.07	14.07%
A	0.28%	-1.80	18.31%
BBB	0.51%	-1.62	22.00%
BB	1.69%	-1.18	34.14%
B	3.34%	-0.88	45.86%

Example calculations: BBB credit rating, first iteration

Iteration	DP Guess	f(a)	f($\hat{a}$)	f'($\hat{a}$)
1	-1.0000	0.0251	0.1215	0.2578
2	-1.3740	0.0251	0.0499	0.1316
3	-1.5630	0.0251	0.0296	0.0859
4	-1.6157	0.0251	0.0253	0.0755
5	-1.6194	0.0251	0.0251	0.0748
6	-1.6194	0.0251	0.0251	0.0748

To apply the Newton-Raphson method so solve for the default point we must start with a guess as to what that default point may be. Our starting point is therefore...

$$\text{Default point guess} = -1.0000 \tag{13}$$

Using Equation (1) above, the cumulative default rate applicable to BBB rated debt is...

$$f(a) = 1 - \text{Exp}\left\{ -0.0051 \times 5.00 \right\} = 0.0251 \tag{14}$$

Using Equations (8) and (12) above...

$$f(\hat{a}) = f(-1.0000) = \int_{-\infty}^{-1.0000} \sqrt{\frac{1}{2 \times \pi \times 0.6125}} \times \text{Exp}\left\{ -\frac{1}{2 \times 0.6125} \times (\theta - (-0.0862))^2 \right\} \delta\theta = 0.1215 \tag{15}$$

Using Equations (9) and (12) above...

$$\frac{\delta f(a)}{\delta a} = \sqrt{\frac{1}{2 \times \pi \times 0.6125}} \times \text{Exp}\left\{ -\frac{1}{2 \times 0.6125} \times (-1.0000 - (-0.0862))^2 \right\} = 0.2578 \tag{16}$$

Using Equation (10) above and the results of Equations (13) to (16) above, our next best estimate of the default point a for iteration two is...

$$-1.3740 = -1.0000 + \frac{0.0251 - 0.1215}{0.2578} \tag{17}$$

After iteration six the estimated default point is -1.6194. Using Equation (11) above, the debt leverage ratio that is consistent with a BBB credit rating is...

$$\Gamma = \frac{1}{0.9000} \times \text{Exp}\left\{ -1.6194 \right\} = 0.2200 \tag{18}$$

References

[1] Gary Schurman, *Credit Ratings - Default Rates, Recovery Rates and Credit Spreads,* June, 2023.

[2] Gary Schurman, *The Newton Raphson Method For Solving Non-Linear Equations,* October, 2009.

[3] Gary Schurman, *The Calculus of the Normal Distribution,* October, 2010.